

# Aspects of the density field in an active nematic

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## Abstract

Active nematics are conceptually the simplest orientationally ordered phase of self-driven particles, but have proved to be a perennial source of surprises. We show here through numerical solution of coarse-grained equations for order parameter and density that the growth of the active nematic phase from the isotropic phase is necessarily accompanied by a clumping of the density. The growth kinetics of the density domains is shown to be faster than the 1/3-law expected for variables governed by a conservation law. Other results presented include the suppression of density fluctuations in the stationary ordered nematic by the imposition of an orienting field. We close by posing some open questions.

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## I. INTRODUCTION

### A. Background: nematic and polar order, active and passive

The nematic [1, 2] is the simplest liquid-crystalline phase, a true fluid in all directions, but with a preferred axis. It possesses uniaxial, fore-aft symmetric orientational order and no positional order. This paper is concerned with systems with this spatial symmetry, but without time-reversal symmetry: each constituent particle of the system is endowed with the ability to consume free energy which it transduces into movement [3, 4]. This ordered phase of self-driven matter [5–7] is known as an active nematic [8].

Early interest [9–11] in flocks as ordered phases of nonequilibrium condensed matter focused naturally on the case where the individual constituents and the emergent ordered state observe the distinction between ahead and behind. A flock, after all, is a collection of creatures that is going somewhere. The idea of classic liquid-crystalline order in living matter, with anisotropy but no polarity, appeared initially in analyses of patterns in aggregates of elongated cells [12], but the idea that such phases were wildly different from their dead thermal-equilibrium counterparts emerged only gradually [5, 8]. The subject has grown and evolved considerably since, with studies on the nature of density fluctuations [13–15], the passage from microscopic to coarse-grained descriptions [16–18] and the discovery and elucidation of intrinsic instabilities in the latter [17, 19], and the unique properties of topological defects in active nematics [13, 20–24]. Clear numerical evidence is now available for the existence of a statistically homogeneous state with nematic order in an noisy active system, in a study [25] that poses new puzzles regarding the scaling properties of density fluctuations.

To set the stage, we review briefly some basic findings from the initial literature on active nematics. Throughout this paper we will consider systems in which the total number of particles is the only conserved quantity. We will assume that all parts of the system under consideration are homogeneously in contact with a passive momentum sink, so that the velocity field of the system relaxes on a finite timescale to a value determined by the slow variables, *viz.*, the number density field  $\rho$  and the traceless, symmetric, second-rank tensor

$\mathbf{Q}$  characterising nematic order [1, 2]. Effects relating to hydrodynamic flow [7, 20–23] will be mentioned in passing, if at all.

## B. Currents from orientational distortions

The first hint of peculiarities in the dynamics and statistics of  $\rho$  in an active nematic came from noting [8] that on general grounds the particle current  $\mathbf{J} = \rho\mathbf{v}$ , which defines the velocity field  $\mathbf{v}$ , is permitted to have a nonequilibrium contribution

$$\mathbf{J}_{act} \propto \rho \nabla \cdot \mathbf{Q} \quad (1)$$

proportional to the orientational curvature. In more detail, following refs. [6–8], the origin of the term can be understood as follows. Consider a general collection of particles, interacting with each other and moving while in contact with a dissipative substrate that serves as a sink for momentum. Balancing friction with the substrate against other force densities  $\mathcal{F}$  arising from interparticle interactions or external driving fields yields  $\mathbf{v} = \mathbf{M} \cdot \mathcal{F}$  where  $\mathbf{M}$  is a mobility – the inverse of a damping coefficient. For the case where the non-frictional forces come entirely from interparticle interactions, and are therefore momentum conserving, we can write  $\mathcal{F} = -\nabla \cdot \boldsymbol{\sigma}$  where  $\boldsymbol{\sigma}$  is the stress tensor. The deviatoric stress for an *equilibrium* nematic system with a free-energy functional  $F$ , to leading order in  $\mathbf{Q}$ , is proportional to  $\delta F / \delta \mathbf{Q}$ , and thus vanishes for the mean state of nematic order. Leading-order stresses for a perturbed nematic involve gradients of the order parameter, weighted by the Frank elastic constants. However, for a nematic steady state away from thermal equilibrium, the order parameter value is not determined by free-energy minimisation, and a piece of the stress proportional to  $\mathbf{Q}$  itself, rather than to its conjugate field, is permitted, leading to the form (1) at leading order in a gradient expansion.

The resulting statistical steady state must then be governed by a dynamic balance between two kinds of mass fluxes:  $\mathbf{J}_{act}$  driven by orientational inhomogeneities and a diffusional current  $\propto \nabla \rho$  ironing out density inhomogeneities. Deep in the nematically ordered phase the dominant contribution to  $\nabla \cdot \mathbf{Q}$  will come from spatial variations in the principal axes of  $\mathbf{Q}$ , not its magnitude. Fluctuations in density should then be comparable in magnitude to those

in the orientation. The latter, in the nematic, which is a phase in which continuous rotation invariance is broken spontaneously, should diverge at small wavenumber. So, therefore, should fluctuations in the number density, by the foregoing argument [8]: regions in an active nematic containing on average  $N$  particles should display fluctuations in the number with a standard deviation growing more rapidly than  $\sqrt{N}$ . Giant number fluctuations have now been seen in experimental [13] and numerical realisations [14] of active nematics, but our understanding of this deceptively simple system continues to evolve [25, 26].

### C. Microscopic model of apolar flocking

We summarise briefly the microscopic model of Chat  , Ginelli and Montagne [14] for active, uniaxial, headless particles with a tendency to align with each other. The particles, labelled by  $\alpha$ , have positions  $\mathbf{R}_\alpha$  and orientations  $(\cos \theta_\alpha, \sin \theta_\alpha)$  with respect to a fixed reference frame. Particles move synchronously at discrete times  $t$  by a fixed distance  $\epsilon$ . in the direction defined by  $\theta_\alpha$  or  $\theta_\alpha + \pi$  with equal probability. The time step is taken as unity, and the updated orientation is defined in two stages: first, orient the particle parallel to the mean of its neighbours, along the direction of the first eigenvector of the average of the individual orientation tensors

$$\mathbf{Q}_\beta = \begin{pmatrix} \cos^2 \theta_\beta - \frac{1}{2} & \cos \theta_\beta \sin \theta_\beta \\ \cos \theta_\beta \sin \theta_\beta & \sin^2 \theta_\beta - \frac{1}{2} \end{pmatrix} \quad (2)$$

of all particles  $\beta$  including  $\alpha$  with  $|\mathbf{R}_\beta - \mathbf{R}_\alpha|$  less than a fixed interaction range, taken to be substantially larger than the individual displacements  $\epsilon$ . Next, add to it a small noise:

$$\theta_\alpha(t+1) \rightarrow \theta_\alpha(t) + \sigma \zeta_\alpha(t) \quad (3)$$

where  $\zeta_\alpha$  is distributed uniformly on  $[\pi, \pi]$ , identically and independently for each  $\alpha$  and  $t$ , and  $\sigma$  controls the noise amplitude.

Coarse-graining the above rules was shown [16, 19] to generate the active current  $\mathbf{J}_{act} \propto \rho \nabla \cdot \mathbf{Q}$  [8] discussed above. The complete PDEs for the number density

$$\rho(\mathbf{r}, t) = \sum_{\alpha} \delta(\mathbf{r} - \mathbf{R}_\alpha(t)) \quad (4)$$

and the nematic tensor density

$$\mathbf{w}(\mathbf{r}, t) \equiv \rho \mathbf{Q}(\mathbf{r}, t) = \sum_{\alpha} \mathbf{Q}_{\alpha} \delta(\mathbf{r} - \mathbf{R}_{\alpha}(t)) \quad (5)$$

were shown [19] to have the general form

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \Delta \rho + \frac{1}{2} \mathbf{\Gamma} : \mathbf{w} \quad (6)$$

and

$$\frac{\partial \mathbf{w}}{\partial t} = \mu \mathbf{w} - 2\xi \mathbf{w}(\mathbf{w} : \mathbf{w}) + \frac{1}{2} \Delta \mathbf{w} + \frac{1}{8} \mathbf{\Gamma} \rho. \quad (7)$$

In (6) and (7)  $\Delta$  is the Laplacian, the coefficients  $\mu$  and  $\xi$  are calculable functions [19] of  $\rho$  and the noise strength  $\sigma$  in (3). For our numerical study we choose a simpler density-dependence of  $\mu$  and  $\xi$  than calculated in [19], as discussed in section (II). The tensor differential operator  $\mathbf{\Gamma}$  has components  $\Gamma_{11} = -\Gamma_{22} = \partial_1 \partial_1 - \partial_2 \partial_2$  and  $\Gamma_{12} = \Gamma_{21} = 2\partial_1 \partial_2$  so that  $\mathbf{\Gamma} : \mathbf{w} = 2\partial_{\alpha} \partial_{\beta} w_{\alpha\beta}$ .

### 1. *Multiplicative noise*

Equations (6) and (7) represent only the mean-field behaviour flowing from the microscopic model. The complete picture must include the noise sources induced in the coarse-grained dynamics by the stochastic part of the microscopic dynamics, as discussed in [16, 19]. The noise in (6) is the divergence of a random current with covariance of order  $\rho(\mathbf{Q} + \mathbf{I}/2)$  where  $\mathbf{I}$  is the unit tensor, while that in (7), rewritten as an equation for  $\mathbf{Q}$ , is of order  $1/\rho$ . Crucially,  $\rho$  and  $\mathbf{Q}$  in the foregoing refer to the *local, instantaneous* values of the fields in question. Such density-dependent multiplicative noise is a generic consequence [27] of the Central Limit Theorem together with the fact that  $\rho$  and  $\mathbf{Q}$  are, respectively, the density of an extensive quantity and the local expression of an intensive quantity. Stochastic-PDE treatments of dynamic critical phenomena at equilibrium [28] implicitly ignore such field dependence, presumably because it is irrelevant there. In active matter where density fluctuations are large across an entire phase rather than just at phase boundaries, it could well have more serious consequences [19], but we will not explore this issue further in this paper. In section(II) we will present results from a numerical solution of (6) and (7), mainly without noise sources.

## 2. Proliferation of topological defects

Topological defects in active nematics pose puzzles that are still to be resolved – and that, regrettably, we shall not resolve in this article. The headless nature of nematic order means the defects have half-integer strength, representing a rotation through  $\pm\pi$  of the order parameter upon circumambulating an elementary disclination [1]. Structurally, defects of strength  $-1/2$  have a 3-fold symmetric appearance while strength  $+1/2$  defects have a polarity and hence move in a directed fashion because active nematics are out of equilibrium [13, 21–23]. In this paper we shall steer clear of issues concerning the degradation of nematic order by defect proliferation. We shall assume defects have a prohibitively high core energy and are thus not produced at all in steady state on accessible length scales. In coarsening studies, however, the onset of local order about the initial statistically isotropic and homogeneous state forces the emergence of defects whose annihilation leads to the growth of ordered domains.

### D. Problems of interest and summary of results

Here we present our results in brief, from a study of the noiseless [29] equations (6) and (7) describing the dynamics of the density and nematic order parameter of an active nematic-forming system. (i) When quenched from the isotropic to the nematic phase, nematically ordered domains grows with time  $t$ , as expected. The growth law over the range explored is slower than  $t^{1/2}$ , and is consistent with the behaviour expected [30] for the growth of an equilibrium nematic. (ii) More interestingly, the *density* displays clumping and coarsening as well. The associated length scale grows more rapidly than the  $t^{1/3}$  normally associated with domain growth for a conserved order parameter [30]. (iii) The domains for density and nematic order parameter both show a characteristic three-armed morphology, with a topological defect of strength  $-1/2$  at the nodal point of the arms. (iv) Deep in the active nematic phase, we impose an aligning field and measure numerically the steady-state density fluctuations in the presence of a small additive noise. We find that for large enough field the fluctuations saturate at large system size, at a value that diverges as the field is taken to zero. The form of the dependence compares well to analytical calculations from a linearised

theory. The controlling role of the field is a clear indicator of the underlying mechanism of giant density fluctuations, namely, mass flux generated by orientational distortions. (v) We note the strong departure from the standard fluctuation-dissipation relation implied by the large density fluctuations: the *response* to a field coupling to the density is normal, the *fluctuations* of the density are anomalous. We also point out an apparent incompatibility between the anisotropic density correlations in the linearised theory of the active nematic [8] and the quasi-long-range character of two-dimensional nematic order.

The remainder of this paper is organised as follows: In section II we present our study of the growth kinetics of the active nematic from an isotropic background. In section III we examine the effect of an aligning field on nematic and density correlations. In section IV we discuss correlation vs response, and the puzzle of anisotropic density correlations. We close with a summary.

## II. COARSENING AND CLUMPING

In this section we present our numerical study of the growth kinetics of an active nematic following a “quench” from an isotropic initial state, as seen through (6) and (7), with the parameter  $\mu$  allowed to depend on the local density:  $\mu(\rho) = \rho - \rho_c$ , where the threshold density  $\rho_c = 0.5$  and the system is prepared in a nearly uniform initial state with  $\mathbf{w} = 0$  and  $\rho$  at each grid point drawn independently from a Gaussian distribution with mean  $\rho_0 = 0.75$  and standard deviation 0.05. The cubic term in (7) is taken to have coefficient  $2\xi = 1$ . The system is thus in an unstable situation with respect to the growth of nematic order, i.e.,  $\mathbf{w}$ . The value of  $\mu(\rho_0)$  lies well beyond the banding instability [17, 19, 24–26] in the regime where the long-time state should be a homogeneous nematic. As we remarked in the Introduction, we do not include the effects of noise in this study. Randomness enters to a limited extent only via initial conditions. Such an approach is adequate when studying the domain growth of equilibrium phases, where it is established [30] that the long-time, large-scale behaviour is dominated by a zero-temperature fixed point. No comparable result is available for the growth kinetics of a nonequilibrium steady state like the active nematic, so our study must

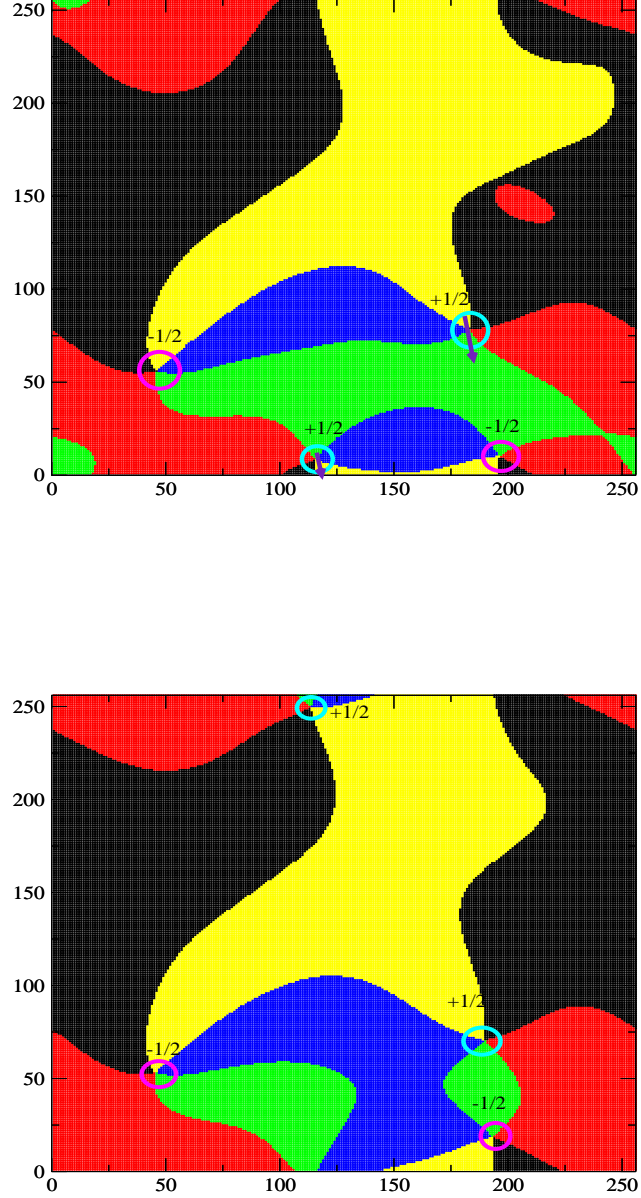


FIG. 1. (color online) Two images of the angle field for nematic order parameter during the coarsening process governed by Equations (6) and (7); the lower figure is at a later time. Colours black-red-green-blue-yellow denote angle ranges  $[0 : 0.2\pi] - [0.2\pi : 0.4\pi] - [0.4\pi : 0.6\pi] - [0.6\pi : 0.8\pi] - [0.8\pi : \pi]$ . The meeting points of different colours are therefore point defects of strength  $\pm 1/2$ .  $+1/2$  and  $-1/2$  defects are marked with red and green circles respectively. Arrows show the direction of motion of  $+1/2$  defects.



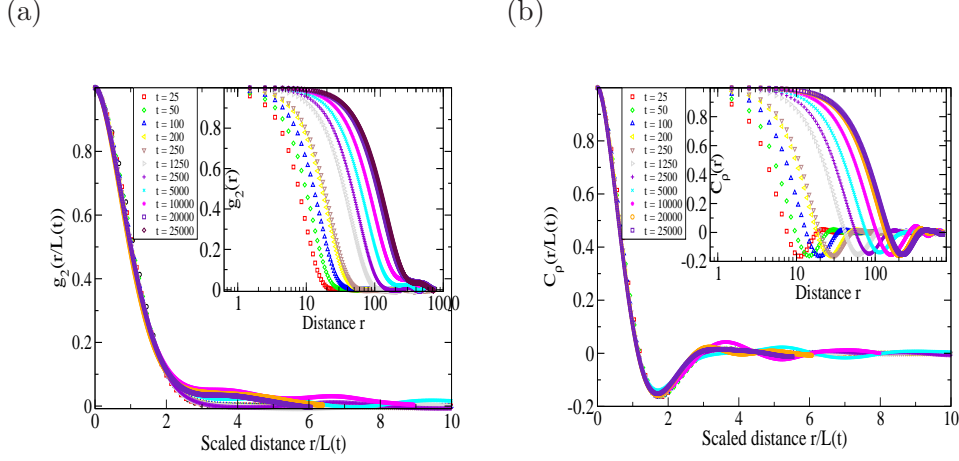


FIG. 2. Evolution of the equal-time two-point nematic order parameter correlator  $g_2(\mathbf{r}, t)$  (a) and density correlator  $C(\mathbf{r}, t)$  (b) with time  $t$ , during domain growth, plotted vs.  $r/L(t)$ , where  $L(t)$  is the distance at which the correlator decays Inset: correlators as functions of the unscaled  $r$

be viewed as a first step. The role of the noise terms [19] in coarsening is currently under investigation.

Following the quench, nematic order appears locally, and the mismatch between the order in neighbouring regions leads to the formation of elementary topological defects or disclinations situated at points in the domain where the magnitude of the order parameter vanishes. If one traces an anticlockwise loop enclosing any one such point, the orientation traverses the range from 0 to  $\pm\pi$  for a defect of strength  $\pm 1/2$ . These are seen in Fig. 1 where the interval  $[0, \pi]$  is divided into five equal colour-coded sectors for visualisation. With the passage of time defects of opposite strength find each other and annihilate, leading to a well-ordered nematic. This is best seen through measurements of the equal-time correlation function  $g_2(r, t) = \text{Tr}\langle \mathbf{Q}(\mathbf{0}, t) \mathbf{Q}(\mathbf{r}, t) \rangle$  of the nematic order parameter  $\mathbf{Q}$ , as a function of spatial separation  $r$  for different times  $t$ , as presented in Fig. 2. Note: the angle brackets in the definition of  $g_2$  as well as that of the density correlator  $C(r)$  below imply an average over all pairs of points separated by a distance  $r$  and over ten initial configurations. We define a characteristic length scale  $L(t)$  as that value of  $r$  for which  $g_2(r)$  decays to 0.1 of its value at  $r = 0$ , and find that graphs of  $g_2(r, t)$  vs  $r/L(t)$  for different  $t$  collapse onto a single curve; see Fig. 2(a).

We will examine below the domain growth law, i.e., the scaling of  $L(t)$  with  $t$ ; let us look first at some interesting features of the morphology. Emanating from each  $-1/2$  defect is a characteristic Y shape, whose three arms are regions with large values of the magnitude of the order parameter  $\mathbf{Q} = \mathbf{w}/\rho$  with one principal axis oriented along the arm. Neighbouring regions outside the Y have much smaller values of  $\|\mathbf{Q}\|$ . Moreover, and this is our central result, the large- $\|\mathbf{Q}\|$  regions are also large- $\rho$  regions (see Fig. 3). The contrast between low and high density regions is as much as a factor of two. Thus, the process of coarsening of nematic order is accompanied by the onset of large inhomogeneities in the density.

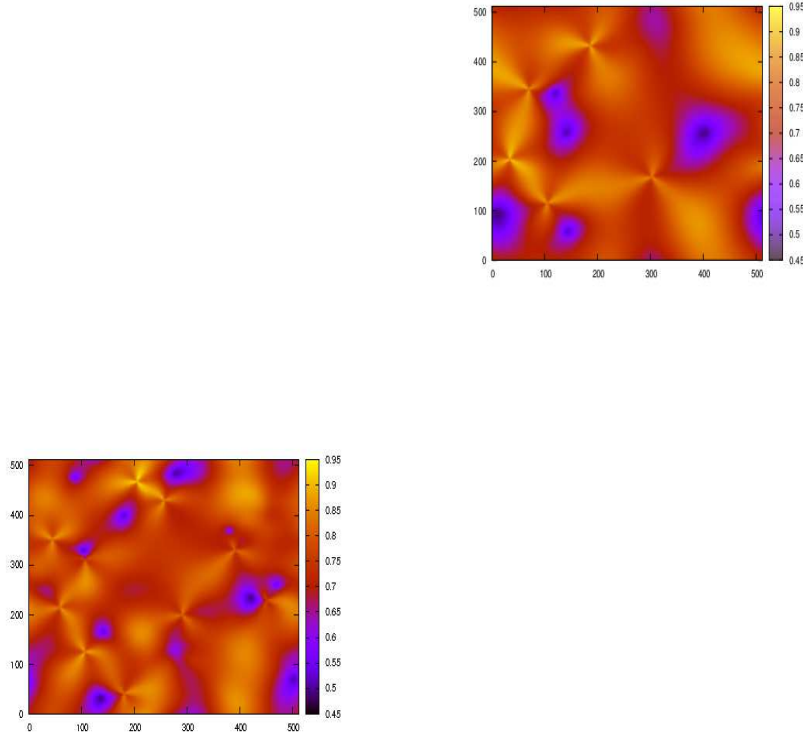


FIG. 3. Density field during the growth of nematic order, from numerical solutions of (6) and (7). Left and right figures show an earlier and a later time configuration. Note the three-armed structures of high density, and the large density variation overall. The order parameter, not shown here, tracks the density – dense regions have high order – and the core of the Y holds a strength  $-1/2$  type defect.

To examine further the nature and evolution of structure in the density field, we measure the two-point density correlation function  $C(r, t) = \langle \delta\rho(\mathbf{0}, t)\delta\rho(\mathbf{r}, t) \rangle$ , where  $\delta\rho(\mathbf{r}, t) = \rho(\mathbf{r}, t) - \rho_0$  is the deviation of the local density from the mean  $\rho_0$ . Like  $g_2$  above,  $C(r, t)$  also coarsens. Again we define a length scale  $L(t)$  as the value of  $r$  at which the function decreases to 0.1 of its value at  $r = 0$ . Fig. 2(b) shows a data collapse when  $C(r, t)$  is plotted as a function of  $r/L(t)$ .

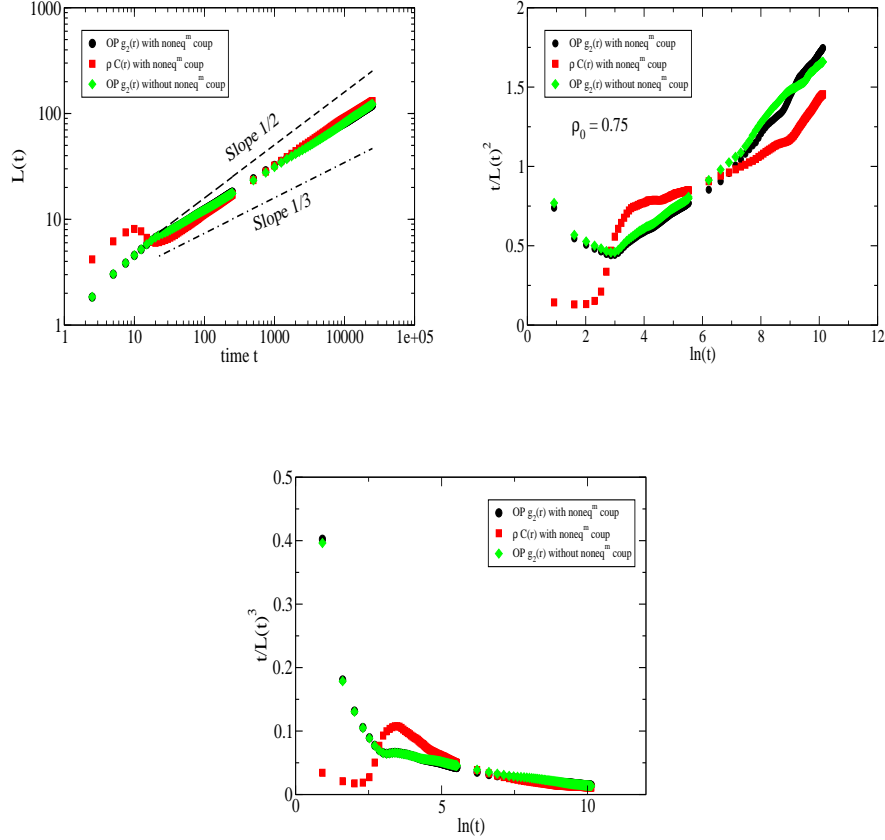


FIG. 4. Upper left: characteristic length  $L(t)$  from normalised two-point correlators of order parameter and density vs. time  $t$  on a log-log scale. Straight lines are with slope 1/2 and 1/3 for comparison. Upper right:  $t/L^2$  vs  $\ln t$ . Lower:  $t/L^3$  decreases consistently with time, suggesting the growth law is distinct from  $t^{1/3}$ .

Equations (6) and (7), which are used for the numerical study, are known to possess a banding instability [17] for parameter values just past the mean-field onset of the nematic

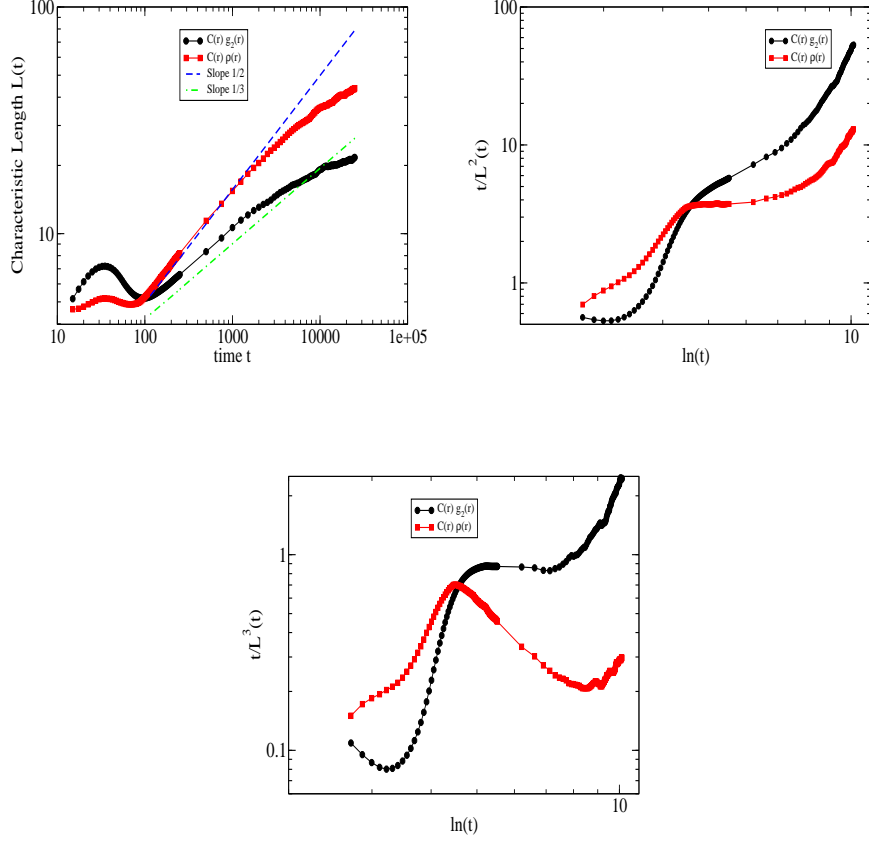


FIG. 5. Upper left: characteristic length  $L(t)$  from normalised two-point correlators of order parameter and density vs. time  $t$  on a log-log scale when we are close to the isotropic-nematic transition, *in the regime of the linear instability to band formation*. Straight lines of slope 1/2 and 1/3 are shown for comparison. Upper and lower right:  $t/L^2$  vs  $\ln t$  and  $t/L^3$  vs  $\ln t$  respectively, suggesting that the density coarsening is faster than  $t^{1/3}$  and consistent with  $t^{1/2}$ , but that nematic ordering is weak or arrested.

phase. To avoid this instability, we study these equations deep in the nematic phase, far from the transition point. The lengths inferred from the coarsening of nematic order and the clumping of the density show essentially the same time dependence (Fig. 4), consistent with a growth law with an exponent slightly smaller than 1/2 (but distinctly larger than 1/3, which shows that  $t/L^3$  decreases steadily with increasing  $t$ ). The upper right frame of Fig. 4 plots  $t/L^2$  vs  $\log t$ ; the roughly linear dependence is consistent – although not definitively

so – with  $L \propto (t/\log t)^{1/2}$ , the law expected for the growth of an equilibrium nematic [30]. For comparison, we also calculate the lengths for coarsening of the nematic order parameter and density when we are in the banding regime (Fig. 5), and find the density still coarsens much as in Fig. 4 but the nematic order fails to keep up, consistent with the approach to a phase-segregated but statistically isotropic state [17, 25].

The reason for the clumping is not far to seek. The nematic orientation field has a three-fold symmetric structure around a  $-1/2$  defect. The resulting curvature, via the term  $1/2\mathbf{\Gamma} : \mathbf{w}$  in (6), derived for the microscopic model [14, 19] as outlined in section IC, drives currents in (6), towards the defect core. This results in a Y-shaped density pattern. The  $\mathbf{\Gamma}$  term in the order-parameter equation (7) promotes anchoring of the nematic orientation with respect to density gradients. The  $\rho$  dependence of  $\mu$  in (7) leads to loss and gain, respectively, of nematic order in the resulting low- and high-density regions. Minor variations in the microscopic model will lead to equations of motion of the same form but with changes in the magnitude and, in principle, the signs of coefficients of the  $\mathbf{\Gamma}$  term discussed above as well as other terms.

Independent of the specific form of the coarsening laws, we emphasise that clumping during the coarsening of an active nematic is a rather remarkable finding. An initial state with uniform density condenses and coarsens spontaneously. The evolution of the density is formally similar to that taking place during phase separation, despite the absence of an attractive interaction or a negative second virial coefficient. Note, in particular, that it is distinct from the giant number fluctuations in a noisy active nematic steady state, although the curvature-induced current term proportional to  $\mathbf{\Gamma}$  in (6) is responsible for both. What distinguishes the process is that it takes place side-by-side with, and as a by-product of, the coarsening of nematic order. Of course in our noise-free system, in a situation where the uniform nematic is linearly stable at all wavenumbers, the ultimate final state must be a quiescent, homogeneous nematic of uniform density. It is important to note that the mechanism driving the clumping in our case is distinct from the banding instability that arises just past the mean-field threshold for the isotropic-nematic transition in an active system [17, 19, 25]. We close this section by noting that a complete description of the final state is incomplete without the inclusion of noise. When noise of the appropriate type [19] is

taken into account, we expect interesting modifications to the correlation functions. We are currently investigating the effect of noise on the correlators during coarsening [31]. At long times, these must cross over to the equal-time correlators of the fluctuating active nematic steady state [8, 25].

### III. AN ORIENTING FIELD SUPPRESSES DENSITY FLUCTUATIONS

The study of [8] says giant number fluctuations (GNF) are a consequence of curvature-induced currents, i.e., that large-scale orientational fluctuations are responsible for the currents that give rise to large density inhomogeneities. Since the cause of GNFs in the steady state is long-wavelength fluctuations in the angle field, we ask what happens if the latter are suppressed by an external aligning agency, as can be accomplished in a conventional liquid-crystal system by imposing a magnetic or electric field. Our findings offer a method to check in principle whether the large density fluctuations in experiments [13] on active nematics are indeed a result of the mechanism of [8]. The idea is simple: if we apply an external field to a nematic, the restoring force in response to an orientational distortion will not vanish even in the limit of a spatially uniform change in direction. The amplitude of orientational fluctuations will then saturate rather than diverge at the smallest wavenumbers. The arguments in section I B then tell us that the density fluctuations will be cut off as well.

To test this idea, let us obtain a convenient form for the density fluctuations in the presence of an orienting field. As the nematic order parameter is a traceless symmetric tensor, such a field  $\mathbf{h}$  imposed on a equilibrium nematic will lead to a term  $\mathbf{Q} : \mathbf{h}\mathbf{h}$  in the governing free-energy density. Note that we absorb details of material parameters involved in such a coupling into a redefinition of  $\mathbf{h}$ , and choose the sign of the coefficient to be positive without loss of generality by exploiting the  $\mathbf{Q} \rightarrow -\mathbf{Q}$  symmetry [1] of two-dimensional nematics, the case of particular interest here. The result is that the equation of motion (7) for the order parameter is modified by the presence of a term  $-\mathbf{h}\mathbf{h} + (h^2/2)\mathbf{I}$  on the right-hand side, favouring a mean state in which one eigendirection of  $\mathbf{Q}$  is along  $\mathbf{h}$ . Perturbations about this mean state relax at a rate that approaches a nonzero value of order  $h^2$  in the

limit of small wavenumber. To assess the effect of this aligning field on density fluctuations requires, strictly speaking the inclusion of a noise in the equations of motion, with nontrivial dependence on the local values of  $\mathbf{Q}$  and  $\rho$  as discussed in [19]. A complete treatment of the stochastic PDEs for this problem, with a “faithful” rendition of the noise is a subject of current study. In order to get a qualitative idea of how a field suppresses density fluctuations, we introduce noise in a minimal manner, only in the orientational dynamics. We represent the nematic order parameter in terms of an amplitude  $S$  and an angle  $\theta$ ,  $\mathbf{Q} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$  with  $(A, B) = (S/2)(\cos 2\theta, \sin 2\theta)$ , and introduce a spatiotemporally white noise  $f_\theta(\mathbf{r}, t)$  with correlator  $\langle f_\theta(\mathbf{0}, 0) f_\theta(\mathbf{r}, t) \rangle = 2\Delta_0 \delta(\mathbf{r}) \delta(t)$ . We work with  $\Delta_0 = 0.03$ , and an aligning field  $\mathbf{h} = (h, 0)$  with  $h^2 = 0.001, 0.005, 0.01, 0.05$  and  $0.1$ . In general, the hydrodynamic equations of motion derived from the microscopic rule have multiplicative noise terms [16, 19]. However, for simplicity, we use additive spatiotemporally white noise. We study the resulting stochastic partial differential equations numerically in the same regime deep in the nematic phase as was explored in the coarsening studies. We compare the measured density correlation functions with approximate analytical expressions from a linearised treatment of small fluctuations about the mean uniform ordered phase. We focus on the equal-time spatial autocorrelator  $S_{\mathbf{q}}$  of the density, as a function of wave-vector  $\mathbf{q}$ , at fixed mean density  $\rho_0 = 0.75$ . We present results only for  $q_x = q_z$ . From the equations of motion presented above, a straightforward calculation within a linearised approximation (presented in Appendix A) will show that, in the presence of a field  $h$ , the standard deviation  $\Delta N$  of density fluctuations for a region with  $N$  particles on average satisfies

$$\frac{(\Delta N)^2}{N} = \mathcal{K} + \frac{N}{(a + Nh^2)^2} \quad (8)$$

where  $\mathcal{K}$  is a finite background coming from local number-conserving processes and  $a$  is a constant. Even in the absence of a noise directly driving the  $\rho$  equation, nonlinear effects in (6) and (7) are expected to lead to such a background, so we retain it when fitting our data. Note that the second term in (8) contains the giant number fluctuations [8] modified by the field  $h$ . The form of that term implies that as one looks at regions of increasing size and hence increasing  $N$ , the excess relative density fluctuations initially grow as they would without a field and then decrease once  $N$  goes past a crossover value of order  $h^{-2}$ . We compare (8), viewed as an expression for the density structure factor at wavenumber

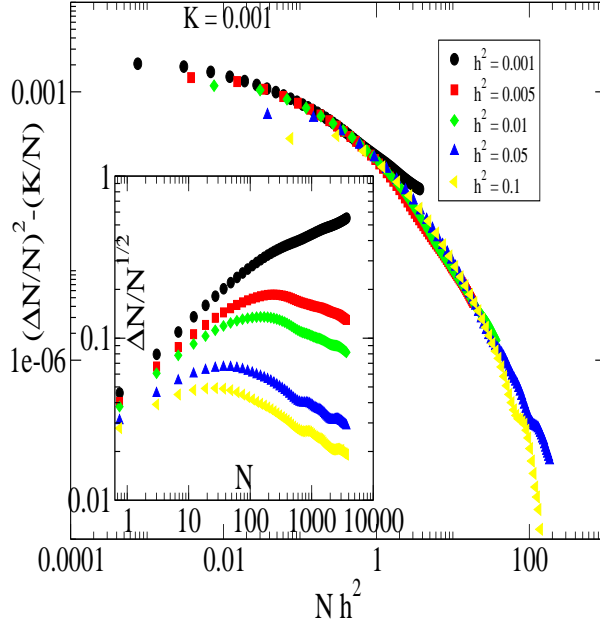


FIG. 6. (color online) An orienting field limits giant number fluctuations: scaled and background-subtracted variance, for different values of field, plotted against scaled mean number, lie on a single curve. Inset: unscaled data, showing the suppression of number fluctuations at large scales in the presence of a field.

$q \sim N^{-1/2}$ , to numerical solutions of (6) and (7) with noise and field. We infer a value of  $\mathcal{K}$  by looking at the smallest wavenumber data in the presence of a large field  $h$ , and subtract that value uniformly from all our data. The resulting quantity, divided by  $N$ , should yield a data collapse when plotted against  $Nh^2$ . Reasonably satisfactory results along these lines can be seen in Fig. 6, with a background subtraction of  $\mathcal{K} = 0.001$ . We emphasise that our aim, from these studies with limited dynamic range, is to underline the role of an externally imposed field as a way of testing the origin of the giant density fluctuations, rather than arguing for a particular value of scaling exponents (see [25]).



## IV. OTHER ISSUES

### A. Correlation vs response

At an *equilibrium* critical point, density fluctuations are large essentially because the response function is large and slowly decaying. This means that if the density at a point is held at a value different from the equilibrium value, there will be a slowly decaying density profile around it. In an active nematic, as can be seen in [8] the fluctuations are large not because of the density propagator, but because orientational fluctuations enter the density dynamics as correlated, nonequilibrium noise. A formal calculation of the response of the density to a change in chemical potential will reveal nothing singular. Is there, then, a field to which the mean density response is large and slowly decaying? The answer is yes, the orientation. Thus, a field perturbing the orientation field will produce long-range distortions in density. In addition, suppressing spontaneous fluctuations of the nematic, by an external field, should kill the giant number fluctuations, as we showed in section III.

### B. The meaning of anisotropic large-scale correlations in an active nematic

In two space dimensions, the setting for most studies referred to in this paper, particle-based simulations [14, 25] (as well as a renormalisation-group (RG) treatment of nonlinearities in the limit where the density field is ignored [32]) find that the active nematic phase is distinguished from the isotropic phase only by the presence of power-law orientational correlations; the order parameter goes to zero algebraically with increasing system size. Concomitantly, the nonequilibrium analogues to the two Frank elastic constants for a two-dimensional (2D) active nematic flow to equality at large length scales under the RG as they do in an equilibrium nematic [33]. Although quasi-long-range order is standard in equilibrium 2D nematics, the active current (1) [8] is explicitly anisotropic when expanded about the ordered nematic phase, with  $x$  and  $z$  components respectively proportional to  $\partial_z\theta$  and  $\partial_x\theta$  for slowly-varying angular fluctuations  $\theta$ . The predicted consequences are *anisotropic* power-law density cor-

relations at small wavevectors  $\mathbf{q} = (q_x, q_z)$ , of the form  $q_x^2 q_z^2 / q^6$ . On the other hand, quasi-long-range order, as known to hold in two-dimensional active nematics as well, implies the absence of a true nematic on the largest length scales, and thus allows no global anisotropy even in the sense of a spontaneously broken symmetry. Thus, just as nonlinearities must cause the system to flow to a one-Frank-constant picture at large scales [32] when the density is ignored, they presumably also isotropise the density correlator. From ref. [25], it would appear that they also change its scaling from anisotropic  $1/q^2$  to isotropic  $1/q^{2-\eta}$ . A renormalisation-group theory of the value of  $\eta$  remains a challenge.

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## APPENDIX A: ACTIVE NEMATIC WITH AN EXTERNAL FIELD

Using the representation  $\mathbf{Q} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$ , where  $(A, B) = \frac{S}{2}(\cos 2\theta, \sin 2\theta)$ , taking the density  $\rho = \rho_0 + \delta\rho$ , where  $\rho_0$  is the mean density, keeping terms to linear order in  $\theta$  and  $\delta\rho$ , and treating  $S$  as constant we get

$$\frac{\partial \delta\rho}{\partial t} = B_1 \partial_x^2 \delta\rho + B_2 \partial_z^2 \delta\rho + D_2 \partial_x \partial_z \theta \quad (9)$$

$$\frac{\partial \theta}{\partial t} = A_1 \partial_x^2 \theta + A_2 \partial_z^2 \theta + D_1 \partial_x \partial_z \delta\rho - h^2 \theta + f_\theta \quad (10)$$

where  $h$  is the strength of the external applied field.  $f_\theta(\mathbf{r}, t)$  is non-conserving spatiotemporal Gaussian random white noise with noise-noise correlation  $\langle f_\theta(\mathbf{r}, t) f_\theta(\mathbf{r}', t') \rangle = 2\Delta_0 \delta(\mathbf{r} - \mathbf{r}') \delta(t - t')$ .  $\Delta_0$  is noise strength for orientation equation. Writing equations (9 and 10) in Fourier space we get

$$(-i\omega + A_1 q_x^2 + A_2 q_z^2 + h^2) \theta(\mathbf{q}, \omega) = -D_1 q_x q_z \delta\rho(\mathbf{q}, \omega) + f_\theta(\mathbf{q}, \omega) \quad (11)$$

$$(-i\omega + B_1 q_x^2 + B_2 q_z^2) \delta\rho(\mathbf{q}, \omega) = -D_2 q_x q_z \theta(\mathbf{q}, \omega) - i\mathbf{q} \cdot \mathbf{f}_\rho(\mathbf{q}, \omega). \quad (12)$$

Solving these two coupled linear equations (11, 12) for  $\theta(\mathbf{q}, \omega)$  and  $\rho(\mathbf{q}, \omega)$ , we see

$$\begin{pmatrix} -i\omega + A_1 q_x^2 + A_2 q_z^2 + h^2 & D_1 q_x q_z \\ D_2 q_x q_z & -i\omega + B_1 q_x^2 + B_2 q_z^2 \end{pmatrix} \begin{pmatrix} \theta(\mathbf{q}, \omega) \\ \delta\rho(\mathbf{q}, \omega) \end{pmatrix} = \begin{pmatrix} f_\theta(\mathbf{q}, \omega) \\ -i\mathbf{q} \cdot \mathbf{f}_\rho(\mathbf{q}, \omega) \end{pmatrix} \\ \Rightarrow \begin{pmatrix} \theta(\mathbf{q}, \omega) \\ \delta\rho(\mathbf{q}, \omega) \end{pmatrix} = \mathbf{M}^{-1} \begin{pmatrix} f_\theta(\mathbf{q}, \omega) \\ -i\mathbf{q} \cdot \mathbf{f}_\rho(\mathbf{q}, \omega) \end{pmatrix} \quad (13)$$

$$\text{where } \mathbf{M} = \begin{pmatrix} -i\omega + A_1 q_x^2 + A_2 q_z^2 + h^2 & D_1 q_x q_z \\ D_2 q_x q_z & -i\omega + B_1 q_x^2 + B_2 q_z^2 \end{pmatrix}$$

and  $\mathbf{M}^{-1} = \frac{1}{\det[\mathbf{M}]} \begin{pmatrix} -i\omega + B_1 q_x^2 + B_2 q_z^2 & -D_1 q_x q_z \\ -D_2 q_x q_z & -i\omega + A_1 q_x^2 + A_2 q_z^2 + h^2 \end{pmatrix}$  Now the quantity in which we are interested is the two-point density correlator, given by

$$\begin{aligned} \langle \delta\rho(\mathbf{q}, \omega) \delta\rho(-\mathbf{q}, -\omega) \rangle &= \langle |\delta\rho(\mathbf{q}, \omega)|^2 \rangle \\ &= \frac{2(\omega^2 + (A(\hat{\mathbf{q}})q^2 + h^2)^2) \Delta_\rho(\hat{\mathbf{q}}) q^2 + 2D_1^2 q_x^2 q_z^2 \Delta_0}{\det[\mathbf{M}] \det[\mathbf{M}^*]} \end{aligned} \quad (14)$$

where  $\det[\mathbf{M}]\det[\mathbf{M}^*] = \omega^2((A(\hat{\mathbf{q}}) + B(\hat{\mathbf{q}}))q^2 + h^2)^2 + ((A(\hat{\mathbf{q}})q^2 + h^2)B(\hat{\mathbf{q}})q^2 - D_1 D_2 q_x^2 q_z^2 - \omega^2)^2$ .

In the steady state one can calculate

$$S(\mathbf{q}, t) = \langle \delta\rho(\mathbf{q}, t)\delta\rho(-\mathbf{q}, t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle |\delta\rho(\mathbf{q}, \omega)|^2 \rangle d\omega \quad (15)$$

which after integration over  $\omega$  and for  $q_x = q_z$  gives

$$\frac{(\Delta N)^2}{N} = \frac{N}{(a + Nh^2)^2} \quad (16)$$

where  $\mathcal{K}$  is a finite background coming from local number-conserving processes and  $a$  is a constant. In the absence of a noise directly driving the  $\rho$  equation, nonlinear effects in (6) and (7) are expected to lead a background, expression for  $\frac{(\Delta N)^2}{N}$  is

$$\frac{(\Delta N)^2}{N} = \mathcal{K} + \frac{N}{(a + Nh^2)^2} \quad (17)$$

so we retain it when fitting our data.